Smith, James T. 1975. Group theoretic characterization of metric geometry of arbitrary dimension. In International Congress for Logic, Methodology and Philosophy of Science 1975, paper II-19, 2 pp.

International Congress for Logic, Methodology and Philosophy of Science. 1975. *Contributed Papers*. London, Ontario: National Research Council of Canada. Contains Smith 1975.

GROUP THEORETIC CHARACTERIZATION OF METRIC GEOMETRIES OF ARBITRARY DIMENSION

J.T. Smith

Department of Mathematics

San Francisco State University

This paper presents a characterization of the group of motions of a metric geometry of arbitrary, not necessarily finite dimension. A metric geometry is a structure that satisfies Hilbert's Incidence Axioms, some simple orthogonality conditions, admits reflections in all points and lines, and satisfies the Three-Reflections Principles. These Principles state that the composition of reflections in three collinear points is itself a point reflection, and the composition of reflections in three concurrent lines in a plane e is itself a reflection in a line in e. The metric geometries thus include the Euclidean, hyperbolic, and elliptic geometries over commutative fields whose characteristics are not two. In [2] the author gave equivalent analytic and synthetic definitions of a metric geometry. The motions of a metric geometry are the transformations in the group $\mathfrak{B}$ generated by the union of the sets $\mathcal{R}$ and $\mathcal{G}$ of its point and line reflections. Ewald [1] has characterized this group as a group $\mathfrak{B}$ generated by the union of two subsets $\mathcal{R}$ and $\mathcal{G}$ whose elements satisfy certain axioms suggested by properties of point and line reflections. In the elliptic case, the set $\mathcal{G}$ is unnecessary because each line reflection is the composition of the reflections in two conjugate points. A characterization of the group of motions of an elliptic geometry as a group $\mathfrak{B}$ with a single generating subset $\mathcal{R}$ has been given by the author in [3].

Since each line reflection is the composition of the reflections in an incident point and hyperplane, the group of motions of a metric geometry is also generated by the union of the sets $\mathcal{R}$ and $\mathcal{H}$ of its point and hyperplane reflections. This is the viewpoint of the present paper: the group of motions of an arbitrary metric geometry is characterized as a group $\mathfrak{B}$ generated by the union of subsets $\mathcal{R}$ and $\mathcal{H}$ whose elements satisfy certain axioms suggested by properties of point and hyperplane reflections. The axioms are chosen to permit maximum use of the results in [3] applied to the group of motions of the elliptic geometry of all orthocomplemented hyperplanes through the point corresponding to an element $O \in \mathcal{R}$. Because of the conciseness of the elliptic axiom system in [3], the axiom system of the present paper is much simpler than Ewald's [1].

Let $\mathfrak{B}$ be a group, $\mathcal{R}$ and $\mathcal{H}$ be sets of involutions in $\mathfrak{B}$, suppose $\mathcal{R}$ and $\mathcal{H}$ are each invariant und[er] all inner automorphisms of $\mathfrak{B}$, and suppose $\mathcal{R} \cup \mathcal{H}$ generates $\mathfrak{B}$. The following axioms state necessary and sufficient conditions that there exist an isomorphism from $\mathfrak{B}$ t[o] the group of motions of some metric geometry, and under this isomorphism the members of $\mathcal{R}$ and $\mathcal{H}$ correspond to the r[e]flections in the points and the orthocomplemented hyperplane[s]. Elements of $\mathcal{R}$ will be denoted by upper case letters O t[o] R ; elements of $\mathcal{H}$, by lower case letters h to n. F[or] $\alpha, \beta \in \mathfrak{B}$, the notation $\alpha|\beta$ means $\alpha\beta = \beta\alpha \;\&\; \alpha \neq \beta$; $\alpha_0|\ldots|\alpha_n$ means that $\alpha_i|\alpha_j$ whenever $i \neq j$. The abbrevi[a]tion $\forall_O h,j\,\Phi$ is used for $\forall h,j[O|h,j \rightarrow \Phi]$; existenti[al] formulas are treated similarly. Axiom 8_n is a schema for $n = 0,1,\ldots$.

AXIOM 1. $\forall O\, \forall_O k,\ell,m[\,k \neq \ell \rightarrow$
$\exists_O n,h[\,m,n|h \;\&\; \forall_O j[\,k,\ell|j \leftrightarrow n,h|j\,]]]$.

AXIOM 2. $\forall O\, \forall_O h\, \exists P[\,OP \neq PO \;\&\; \forall_O j[\,P|j \leftrightarrow j|h\,]]$.

AXIOM 3. $\forall O,P,Q[\,O \neq P \rightarrow \exists R,h[\,Q,R|h \;\&\; \forall j[\,O,P|j \leftrightarrow R,h|j]]$

AXIOM 4. $\exists O,h,j[\,O|h,j \;\&\; h \neq j\,]$.

AXIOM 5. $\forall h \exists O[\,O|h\,]$.

AXIOM 6. $\forall O,P,m[\,\forall j[\,O,P|j \rightarrow m|j\,] \rightarrow \exists n\; OPm = n\,]$.

AXIOM 7. $\forall O\, \forall_O k,\ell,m[\,\forall_O j[\,k,\ell|j \rightarrow m|j\,] \rightarrow \exists_O n\; k\ell m = n\,]$.

AXIOM 8_n. $\forall O\, \forall_O h_0,\ldots,h_n[\,h_0|\ldots|h_n \;\&\; O \neq h_0\ldots h_n \rightarrow$
$\exists_O h_{n+1}\; h_0|\ldots|h_{n+1}\,]$.

To see the intuitive basis for these axioms note first that in a metric geometry the correspondence between a point P and the reflection in P is one to one; similarly, the correspondence between an orthocomplemented hyperplane h an[d] the reflection in h is one to one. Point reflections commute if and only if the points are equal or conjugate. Hyperplane reflections commute if and only if the hyperplanes are equal or orthogonal. The reflection in a point P commutes with the reflection in a hyperplane h if and only if P lies on h or P is the pole of h. In the former case th[e] composition of the reflections is the reflection in the perpendicular to h at P; in the latter case, the reflection coincide.

Axiom 1 is a version of Conjugacy Axiom C1 of [3]. (Axiom C1 says that for any points K , L and M with $K \neq L$ in an elliptic geometry there exist points N and H such that M and N are conjugate to H and the lines KL and MN coincide--see Figure 1.) Axiom 1 is Axiom C1 applied to the reflections in hyperplanes through the point corresponding to O .

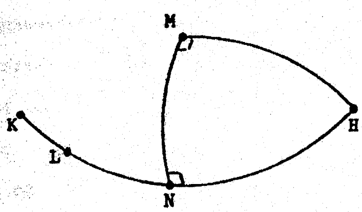

Figure 1. Conjugacy Axiom C1 of [3].

Axiom 2 says that a line OP perpendicular to the hyperplane h can be erected at O . Axiom 3 says that a hyperplane h orthogonal to the line OP can be dropped from Q --see Figure 2. (Axiom 3 is yet another form of the Conjugacy Axiom C1.) Axioms 6 and 7 are forms of the Three-Reflections Principles. Axioms 8_n insure that, in the finite dimensional case, the point reflections are the compositions

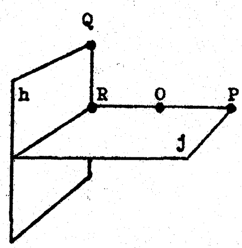

Figure 2. Axiom 3.

of all the reflections in a maximal orthogonal family of hyperplanes.

From a structure $(\mathfrak{B}, \mathfrak{R}, \mathfrak{H})$ satisfying Axioms 1 to 7, $8_1, 8_2, \ldots$ is constructed a metric geometry in the following way. First, for each O the group $\mathfrak{B}_O$ generated by $\{h : O|h\}$ satisfies the axioms of [3] for the group of motions of an elliptic geometry. This dual geometry over O is constructed following [3], then "inverted", yielding the local geometry over O . The "points" of the dual geometry are the "hyperplane reflections" $h \in \mathfrak{H}$; the "conjugacy" relation is $|$. The "points" of the local geometry are the "lines" $[O,h] = \bigcap \{j : O|j|h\}$ (informally speaking). The local geometries are of course elliptic; the conjugacy relation is "orthogonality": $[O,h]$ and $[O,h']$ are conjugate if $h|h'$.

With little trouble the lines and planes connecting the local geometries can be defined; the resulting group geometry satisfies the Incidence and Orthogonality Axioms for metric geometries. The inner automorphisms of $\mathfrak{B}$ corresponding to the elements of $\mathfrak{R}$ and $\mathfrak{H}$ are reflections in the points and orthocomplemented hyperplanes of the group geometry, which is then a metric geometry by the Three-Reflections Axioms 6 and 7. The inner automorphism group of $\mathfrak{B}$ is thus the group of motions of the group geometry. A long argument involving the classification of these motions, using Axioms $8_1, 8_2, \ldots$, shows that the center of $\mathfrak{B}$ is trivial, hence $\mathfrak{B}$ is isomorphic to the group of motions of the group geometry.

To characterize the group of motions of a metric geometry of finite dimension $n \geq 2$ it is only necessary to use Axioms 1 to 3, 5 to 7, and the axiom

$$\exists\, O, h_0, \ldots, h_{n-1} [\, O|h_0|\ldots|h_{n-1} \;\&\; O = h_0 \ldots h_{n-1}] .$$

For the group of motions of an infinite dimensional metric geometry, use Axioms 1 to 3, 5 to 7, and for $n = 0,1,\ldots$ the axioms

$$\exists\, O, h_0, \ldots, h_n [\, O|h_0|\ldots|h_n] .$$

References

1. G. Ewald, "Spiegelungsgeometrische Kennzeichnung euklidischer und nichteuklidischer Räume beliebiger Dimension," Abhandlungen aus dem mathematischen Seminar der Universität Hamburg, XLI (1974), 224-251.

2. J.T. Smith, "Metric geometries of arbitrary dimension," Geometriae dedicata, II (1973), 349-370.

3. J.T. Smith, "Group theoretic characterization of elliptic geometries of arbitrary dimension," will appear in Mathematische Nachrichten.